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# Two-material optimal design for nonlinear elastica☆

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ABSTRACT

The work considers an optimal design problem in the context of nonlinear elastica. More specifically, we deal with finding the best way of mixing fixed amounts of two different elastic materials, so as to minimize the tip deflection of a cantilever beam loaded on its free extreme under the assumption of large deflections. Applying an optimality criteria method to the relaxed problem, simulations give us numerical evidence that the original design problem admits classical solutions (i.e. there is no microstructure) and those are the same as the respective ones for the case of small deflections.



## 1. Nonlinear elastica revisited

Large deflections in flexible beams have constituted a classical problem in nonlinear solid mechanics investigated by many researchers for many years. The first two studies were Bernoulli and Euler in the 17th century, and as a result of that study there arose the Bernoulli–Euler law, an equation that establishes a proportional relation between the curvature at any point of a beam and the applied moment at that point. Such a relationship for initially straight beams is given by

$$\frac{d\theta}{ds} = \frac{M(s)}{EI}, \tag{1}$$

where $\frac{d\theta}{ds}$ is the curvature of the beam, that is, the variation of $\theta$ (angle of rotation of the elastica or deflection curve) with respect to $s$, which is the curvilinear coordinate, measured along the arc length described by the elastica. The term $EI$, known as the flexural stiffness, is the product of the Young's modulus, $E$, and $I$, the moment of inertia of the cross-sectional area of the beam with respect to the bending axis, and $M(s)$ is the bending moment as a function of $s$. In rectangular coordinates, Eq. (1) is expressed as

$$\frac{y''(x)}{(1+y'(x)^2)^{\frac{3}{2}}} = \frac{M(x)}{EI}, \quad x \in (0, L-\Delta), \tag{2}$$

where now both the curvature and the bending moment appear as a function of $x$, the projection of $s$ on the $x$-axis, as we can see in Fig. 1. In most engineering problems the slope of the beam is considered small, which means that we can neglect the square of the first derivative in the curvature formula ($y'(x)^2 \ll 1$), and therefore, use the commonly linearized expression of the curvature instead, that is,

$$y''(x) = \frac{M(x)}{EI}, \quad x \in (0, L),$$

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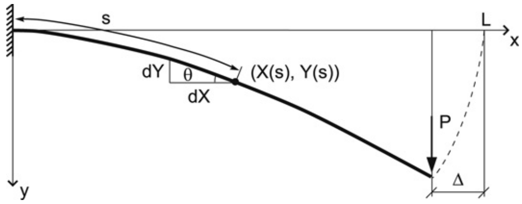



**Fig. 1.** Large deflection of a tip-loaded cantilever beam.

(in this situation $\Delta$ is assumed to be zero). This approach (or the linear model) is justified whenever the deflections are very small compared with the length of the beam, that is, when one has very flat elastica. However, for slender beams (also wires) this simplification is no longer valid because, first, deflections are of the same order of the length of the beam and, second, for large finite loads, deflections are greater than the length of the beam on using the aforementioned linear model. In such cases the underlined geometric nonlinearity is perfectly captured by the model described in Eq. (2).

Eq. (2) is not suitable for describing the nonlinear model when we need to impose specific boundary conditions, as the right end-point is undetermined when we work with rectangular coordinates. The common strategy is to use the infinitesimal angle $\theta$ as the new state variable. Taking the clamped extreme as the origin of coordinates, the bending moment at the free extreme is

$$M = EI\frac{\mathrm{d}\theta}{\mathrm{d}s} = P(L - x - \Delta),$$

where $P$ is the tip-load value, and differentiating in (1) with respect to $s$ we get the equation of the nonlinear beam in terms of the curvilinear angle

$$(EI\theta_s)_s = -P\frac{\mathrm{d}x}{\mathrm{d}s} = -P\cos\theta.$$

How to solve the nonlinear elastica problem has been an object of investigation through the years for a number of researchers. A solution for the nonlinear elastica of cantilever beams of linear elastic material subjected to one vertical load on its free extreme was obtained by Barten [1] and by Bisshopp and Drucker [2] in terms of elliptical integrals. Seames and Conway [3] developed a numerical method for calculating the large deflection of straight and curved cantilever beams for both concentrated and distributed loading. Rodhe [4] studied the case of uniformly distributed load by using power series. Sato [5] did the same for circular cantilever beams. Wang, Lee and Zienkiewicz [6] proposed a numerical method for calculating large deflections of beams for different boundary conditions. Holden [7] obtained the numerical solution for different situations by using a fourth-order Runge–Kutta method.

The references commented on above are only some of the more relevant ones that we can find related to the topic. To get an excellent overview of the problem itself, the reader is also referred to the book on flexible bars [8] by Frich-Fay. More recent numerical approaches are found in [9], where numerical results are corroborated with experiments in the laboratory, and in [10], where a quasi-linear finite difference scheme is developed for large deflection analysis of cantilever beams for different loading conditions. In [11] one can find a real application where a microbeam is working in the nonlinear regime under electrostatic loading.

## 2. Statement of the optimal design problem

The aim of this work is to study the following optimal design problem in the context of the nonlinear elastica: given two materials with different Young's moduli, $E_1$ and $E_2$ ($E_1 \le E_2$), decide how to mix them in a tip-loaded cantilever beam of $L$ length in order to minimize the tip deflection, $Y(L)$, when the amounts of both materials are prescribed beforehand. Taking into account that $\mathrm{d}Y = \sin\theta\mathrm{d}s$ (see Fig. 1), we reformulate the design problem as

$$\min_{\chi} \int_0^L \sin\theta\mathrm{d}s,$$

subject to the state equation

$$\begin{cases}(E(\chi(s))I\theta_s)_s + P\cos\theta = 0, & s \in (0, L),\\ \theta(0) = (E(\chi(s))I\theta_s)\,|_{s=L} = 0,\end{cases}$$

and the volume constraint

$$\frac{1}{L}\int_0^L \chi(s)\mathrm{d}s = V_0, \quad V_0 \in (0, 1).$$

In the state equation the Young's modulus is given by the expression

$$E(\chi(s)) = E_1\chi(s) + E_2(1 - \chi(s)),$$

where $\chi$ is the characteristic function of the measurable subset where the material of constant $E_1$ is placed and whose measure is exactly $V_0$, a fixed and given value.

It is well known that we cannot expect our optimal design problem to admit optimal solutions. See the reference [12] where we may find counterexamples even for the one-dimensional situation. In such a situation, that is to say, when an optimization problem lacks optimal solutions, relaxation is required for it. A relaxed formulation of an optimization problem is a new optimization problem that will admit optimal solutions and such that the optimal solutions can be obtained as the limit, in an appropriate sense that will depend on the nature of the problem, of optimizing sequences of the original problem. The minimum value of the relaxed formulation and the infimum of the original problem must coincide as well.

We are aware of just one reference dealing with an optimal design problem for nonlinear elastica. In the remarkable paper [13] the problem of minimizing the maximum deflection of the nonlinear cantilever elastica of variable cross-section, for loading at one end, is solved. The design variable in that situation is the thickness of the beam, which varies on a fixed interval so that the problem is well-posed and needs no relaxation. In the problem studied in this note we also want to minimize the maximum deflection of a cantilever beam, but of fixed cross-section and made of two distinct materials, our design variable being the distribution of those materials. Another interesting reference in geometrically nonlinear modeling is [14], where topology optimization is used as a synthesis tool for the design of large-displacement compliant mechanisms.

The stress of this note is placed on the fact that optimal design problems for systems governed by nonlinear laws (even geometrically nonlinear) has not been considered before in the mathematical literature. We believe that the problem analyzed is original and interesting from both a mathematical and a practical point of view (flexible robotics for instance), although standard techniques for the linear case work out properly for this geometrically nonlinear one-dimensional model. We do not think that this will be the case in higher dimensions (and we expect to address this issue in the near future).

The layout of the work is the following. Section 3 is devoted to studying the case of small deflections, for which we recover the linear model; in this situation we are able to prove analytically that the problem admits classical solutions. In Section 4, we obtain a relaxation for our design problem in the nonlinear case and carry out sensitivity analysis. Finally, in Section 5, we show some numerical simulations of the optimal design. Those simulations are performed by an optimality criteria method.

## 3. Optimal solutions for small deflections: The linear case

Under the assumption of small deflections, we can assume that $\sin\theta$ can be replaced by $\theta$ and $\cos\theta$ by 1. Then, the design problem is now formulated as

$$\min_{\left\{\chi\in L^\infty((0,L);\{0,1\}):\frac{1}{L}\int_0^L\chi(s)\mathrm{d}s=V_0\right\}}\int_0^L\theta\,\mathrm{d}s, \tag{3}$$

subject to the state equation

$$\begin{cases}(E(\chi(s))I\theta_s)_s+P=0, & s\in(0,L),\\ \theta(0)=(E(\chi(s))I\theta_s)|_{s=L}=0.\end{cases}$$

Now we recover the linear theory for the Euler–Bernoulli beam. In this situation the optimal design problem is particularly simple, and we are able to prove the existence of one optimal design which is classical, that is to say, it takes exactly the values 0 and 1. The proof is fairly simple and elementary and it is described below.

It is elementary to check that the solution for the state equation is given by

$$\theta(s)=P\int_0^s\frac{(L-t)}{E(\chi(t))}\mathrm{d}t. \tag{4}$$

Substitution of (4) in (3) leads to

$$\int_0^L\left(P\int_0^s\frac{(L-t)}{E(\chi(t))}\mathrm{d}t\right)\mathrm{d}s=P\int_0^L\int_t^L\frac{(L-t)}{E(\chi(t))}\mathrm{d}s\mathrm{d}t=P\int_0^L(L-t)^2\left(\frac{1}{E_1}\chi(t)+\frac{1}{E_2}(1-\chi(t))\right)\mathrm{d}t,$$

where we have interchanged the order of integration.

With this rewriting of the optimal design problem it is obvious that if we take the characteristic function, $\chi$, in a subinterval $I_L\subset[0,L]$ of measure $V_0$ such that

$$\min_{t\in I_L}(L-t)^2\geq\max_{t\in(0,L)\setminus I_L}(L-t)^2,$$

then such a $\chi$ will be an optimal design or layout of the materials. It is trivial to check that the only characteristic function verifying this is given by

$$\chi(s)=\begin{cases}1, & s\in[0,V_0L],\\ 0, & s\in(V_0L,L].\end{cases}$$

With this proof we have shown that there is a unique classical solution, but this does not rule out the possibility of the existence of 'relaxed' solutions (see the next section).

## 4. Relaxation and sensitivity analysis

Before going into the analysis of relaxation of the optimal design problem we introduce the dimensionless quantities

$$x = \frac{s}{L}, \quad \lambda = \frac{PL^2}{\alpha E_1 I},$$

where $\alpha \in (0, 1)$ is the ratio $\alpha = \frac{E_2}{E_1}$, and $\lambda$ gives us a measure of the nonlinearity (the more slender the beam, the higher the value of $\lambda$ as well). Then the optimal design problem is rewritten as

$$\min_{\left\{\chi \in L^\infty((0,1);\{0,1\}) : \int_0^1 \chi(x)\mathrm{d}x = V_0\right\}} \int_0^1 \sin\theta(x)\mathrm{d}x,$$

subject to the state equation

$$\begin{cases} \left[\left(\frac{1}{\alpha}\chi + (1-\chi)\right)\theta_x\right]_x + \lambda\cos\theta = 0, & x \in (0,1), \\ \theta(0) = 0, \quad \left(\frac{1}{\alpha}\chi + (1-\chi)\right)\theta_x\Big|_{x=1} = 0. \end{cases}$$

Relaxation for optimal control or optimal design problems for ODE's is a classical subject. In this sense our optimal design problem fits into this framework and relaxation for it is straightforward, by application of Fillipov's theorem, and so proving the relaxed formulation stated below is standard (basic references on the subject, among many others, are [15,16]). Another way to prove the result is by using $H$-convergence [17] and $G$-closure techniques [18]; these were developed for the PDE context and are applicable to our situation, although unnecessary. The relaxed formulation for our optimal design problem will be given by

$$\min_{\left\{\rho \in L^\infty((0,1);(0,1)) : \int_0^1 \rho(x)\mathrm{d}x = V_0\right\}} \int_0^1 \sin\theta\mathrm{d}x,$$

subject to

$$\begin{cases} (a(\rho(x))\theta_x)_x + \lambda\cos\theta = 0, & x \in (0,1), \\ \theta(0) = a(\rho(x))\theta_x|_{x=1} = 0, \end{cases}$$

and $a(x)$ is given by

$$a(\rho(x)) = \frac{1}{\alpha\rho(x) + (1-\rho(x))}.$$

Note that the set of admissible designs has been enlarged to the functions $L^\infty((0,1);(0,1))$ with the same volume constraint, and further the definition of the coefficient of the second-order term of the state equation has been extended in the way usual in this kind of one-dimensional problems. Unlike in the linear case, in this situation we are not able to show existence of classical solutions in a direct manner and we need to carry out a relaxation procedure to get a well-posed problem.

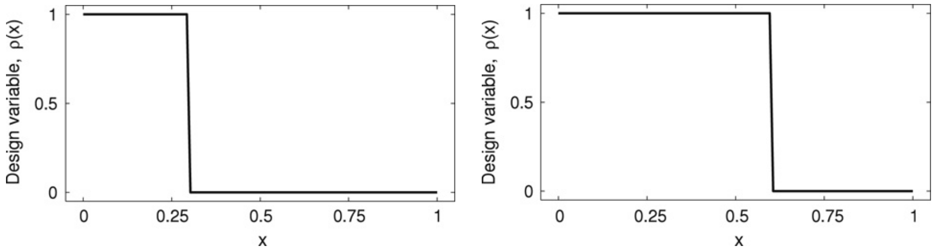
By using the adjoint method we can calculate optimality conditions.

**Proposition 4.1.** *Given $\alpha$, $\lambda$ and $V_0$, there exist measurable functions $c_1$, $c_2$ and a constant $c_3 \in \mathbb{R}$ such that the optimal design $\rho$ satisfies*

$$\begin{aligned} &(1-\alpha)a(\rho(x))^2\theta_x p_x = c_1(x) - c_2(x) + c_3, \\ &c_1(x)\rho(x) = 0, \quad c_1(x) \geq 0, \rho(x) \geq 0, \\ &c_2(x)(\rho(x) - 1) = 0, \quad c_2(x) \geq 0, \rho(x) \leq 1, \end{aligned}$$

*where $\theta$ is the state associated with $\rho$ and $p$ its co-state, i.e. the unique solution of*

$$\begin{cases} (a(\rho(x))p_x)_x - \lambda(\sin\theta)p + \cos\theta = 0, & x \in (0,1), \\ p(0) = a(\rho(x))p_x|_{x=1} = 0. \end{cases}$$




(a) $V_0 = 0.3$. (b) $V_0 = 0.6$.

**Fig. 2.** Optimal solutions.

## 5. Numerical approach and examples

The numerical algorithm used works in a simple way: given an admissible control $\rho_k$ at iteration step $k$, we solve the state equation to obtain $\theta_{x,k}$, and, using both, we get $p_{x,k}$. Keeping in mind that for intermediate controls $(0 < \rho_k < 1)$, $c_1 = c_2 = 0$, we choose as iterative sequence $\rho_{k+1} = \max(10^{-3}, \min(0.99, G_k\rho_k))$ to update the control, where $G_k = \frac{(1-\alpha)a(\rho_k)^2\theta_{x,k}p_{x,k}}{c_{3,k}}$ (notice that when $G_k \to 1$, then $\rho_{k+1} \to \rho_k$). Prior to this we must calculate the value of the multiplier $c_{3,k}$ by using, for example, a bisection method to ensure that $\rho_{k+1}$ is an admissible control at iteration step $k + 1$.

In all the numerical examples we have noticed that the design variable $\rho$ only takes the values 0 and 1 at the end of the optimization process, whatever the values of $\alpha$, $\lambda$, and $V_0$. For the sake of brevity, we illustrate our approach through only two examples (see Fig. 2). In both of them, $\alpha = 0.01$, $\lambda = 4$ and the volume fraction is $V_0 = 0.3$ and $V_0 = 0.6$, respectively.

Judging by the numerical results we can conclude that there is an optimal solution with no microstructure among the two initial materials and, therefore, this lets us conjecture that the original design problem admits optimal solutions in terms of characteristic functions, again without ruling out the possibility of the existence of relaxed solutions.

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